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THIN WALLED BEAMS WITH INTERNAL UNBONDED CABLES: BALANCE CONDITIONS AND STABILITY

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Abstract—A technique for controlling the stress and strain state of beams consists in introducing stretched slipping cables at their interior. This provides a coupling between the local cable strain and the global rod deformation which makes conventional rod models based on local balance conditions inapplicable. The paper presents a model for this kind of structural system in order to formulate the elastic problem and to analyse the stability of known balanced configurations. A generic cable path crossing the interior of the rod is considered. Both variational formulation and local formulation are reported; in particular the latter leads to integro-differential equations. Some qualitative aspects related to the particular coupling between the system components and the effect on the cable path geometry on the stability are discussed. © 1997 Elsevier Science Ltd.

NOTATION

α = slip function
$\boldsymbol{\beta}$ = resultants bi-moment vector for the external forces
$\bar{\boldsymbol{\beta}}$ = bi-moment vector equivalent to prestressing
δ = cable-tube slip
$\boldsymbol{\Gamma}$ = axis deformation descriptor (eqn 9b)
ζ = cross-section co-ordinate
κ = external load multiplier
λ = length of cable path in the deformed configuration
Λ = initial length of the cable path
$\boldsymbol{\mu}$ = vector of resultant bi-moments
$\boldsymbol{\upsilon}$ = vector of resultant bi-shears
ρ = material co-ordinate of the cable point
τ = cable stress
τ_0 = initial cable stress
$\boldsymbol{\varphi}$ = vector of the cross section rigid rotation
$\boldsymbol{\Phi}$ = tensor of the cross section rigid rotation
$\boldsymbol{\chi}$ = vector collecting the warping intensity functions
$\boldsymbol{\psi}$ = vector collecting the warping function
ω = sectorial area
a = cable pure strain
$\mathbf{a}_\gamma$ = unit vectors in the local reference system of the cross section for thin walled beams
A = cross section area
$\mathbf{A}_i$ = unit vectors of the global reference system
$\mathscr{B}$ = beam
$\mathbf{b}_s$ = contact forces
$\mathbf{b}_v$ = volume force
b_i = length of the i-th branch constituting the thin walled beam profile
$\mathscr{C}$ = cable
c = cable stiffness
C = inertial quantities related to the warping function derivatives
$\mathbf{C}$ = beam material stiffness
$\mathscr{D}$ = compatibility formal differential operator
$\mathscr{D}^*$ = equilibrium formal differential operator
$\mathbf{d}$ = strain vector
D = beam cross section
∂D = boundary cross section
$\mathscr{E}$ = euclidean space
e = stretching of the cable
E = normal Young's modulus for the beam
$\mathbf{f}_0$ = cross section deformation descriptor (eqn 9a)

$\mathbf{F}$ = gradient of the deformation $\mathbf{p}$
$\mathscr{G}$ = compatibility formal differential operator
G = shear Young's modulus for the beam
$\mathbf{G}$ = unit vector tangent to the cable path
$\mathscr{H}$ = tube
$\mathbf{h}$ = tube deformation
$\mathbf{H}$ = tube path
I = second grade inertial quantities of the cross section
$\mathbf{I}$ = identity tensor
J = DSV torsional moment of inertia
$\mathbf{K}$ = stiffness matrix for the beam element
$\mathscr{L}$ = linear operator between generalised internal forces (eqn. 50)
L = beam length
$\mathbf{m}$ = resultant moment of the external loads
$\bar{\mathbf{m}}$ = moment equivalent to prestressing
$\mathbf{M}$ = resultant moment of internal stresses
n = transversal co-ordinate of the cross section in the local reference system
$\mathbf{n}$ = resultant of the external loads
$\bar{\mathbf{n}}$ = force equivalent to prestressing
$\mathbf{N}$ = resultant of the internal stress
$\mathbf{p}$ = beam deformation
P = beam material point
$\mathbf{P}$ = beam undeformed configuration
$\mathbf{Q}$ = sectional generalised force vector
$\mathbb{R}$ = real number set
$\mathbf{r}$ = cable strain
$\mathbf{R}$ = cable underformed configuration
$\mathscr{S}$ = mean surface of the beam wall
s = curvilinear abscissa of the cross section profile
S = first grade inertial quantities of the cross section
t = wall thickness of thin walled beams
$\mathbf{t}_i$ = traction on plane $\mathbf{A}_i$
$\mathbf{T}$ = Cauchy's stress tensor
$\mathbf{u}$ = beam displacement field
$\mathbf{u}_0$ = displacement of beam axis
$\mathbf{U}$ = gradient of displacement field
$\mathbf{v}$ = generalised displacement vector
V = region occupied by the beam in the reference configuration
$\mathbf{W}$ = matrix of the stress geometrical effects (eqns 62 and 66)
x_i = co-ordinate of the cross section point.

1. INTRODUCTION

The technique for controlling the stress and strain fields in beams by means of stretched slipping cables is acquiring more and more interest in structural engineering. This technique may be realised mainly in two ways: through lubrificated cables which slip in protective tubes disposed along a prefixed path inside the beam (unbonded prestressing) or through cables disposed outside the beams along paths defined by saddle points (external prestressing). In both the cases the friction between beam and cable is negligible and the stress state provided by the external loads on the cable is related to the deformation of the whole beam so that the conventional local beam model cannot be applied in the analysis.

A large part of literature on this topic reports experimental data or formulations applicable to very special cases only (for more detailed technical literature see Saadatmanesh *et al.* (1992) and Alkhairi *et al.* (1993)). A more general analysis based on a coherent description of system kinematics has recently been presented (Dall'Asta (1996)) in the context of finite deformation theory. This last work is however dedicated to elastic three-dimensional solids and, although it shows some qualitative peculiarities of the problem, it does not investigate many aspects of remarkable technical interest.

The scope of the present paper is to analyse the system behaviour, according to the small deformation theory, when the three-dimensional body consists of a prismatic solid (beam). This particular problem is of evident interest in structural mechanics and the present work intends to furnish a tool for the analysis of numerous aspects which were not clarified in previous studies. In particular, the authors intend to describe the relations arising between the geometry, the stress and strain of the cable and the classical beam stress resultants. Furthermore, it is necessary to formulate and analyse the problem of the equilibrium stability of the system, in order to investigate and clarify some buckling phenomena previously observed in experiments (Saadatmanesh *et al.* (1992)) and in simplified models (Dall'Asta (1996)). Since the problem is particularly important for beams with low torsional stiffness the analysis is based on a rod model suitable for describing the behaviour of beams with open thin-walled cross sections.

In the sequel, the description of kinematics is obtained by interpreting the interaction between cable and beam as a global kinematics constraint exerted by the deformation of the beam on the deformation of the cable. The cross sections of the beam are undeformable on their own plane and the formulation refers to internal cables. The balance condition is obtained from the variational formulation proposed in (Dall'Asta 1996) and the integro-differential equations expressing the local equilibrium are finally obtained. The integral part arises as a consequence of the coupling between local and global deformations. The problem of stability is approached by starting from the linearization of the non-linear equilibrium condition in the neighbourhood of a non natural balanced configuration, introducing the kinematics model adopted. A synthetic formulation has been achieved by evidencing some formal analogies between the internal virtual work of the beam, related to classical resultants, moments, bi-shears and bi-moments, and the functional expressing the admissible variations of the cable strains. This has also simplified the description of the relations existing between the stress resultants of the beam and the interaction forces produced by the cable along its path. Furthermore, a qualitative analysis of the stability condition has permitted showing the role played by the cable on the stability of the system. In particular, it is shown that the effect of the cable is related to geometric quantities only and a criterion for identifying the stabilising cable paths is reported.

2. KINEMATICS OF THE CABLE-BODY SYSTEM

2.1. *Reference configuration*

The considered structural system consists of a beam with rectilinear axis and a cable; the latter is anchored at its ends and is free to slip along a path which traces a generic curve in the interior of the beam.

The beam $\mathscr{B}$ is a three-dimensional manifold of the euclidean space $\mathscr{E}$. In the reference configuration its material points occupy the cylindrical region V, for which a convenient parametrization is obtained by introducing an orthonormal reference system with basis $\{\mathbf{A}_i; i = 1, 2, 3\}$. The beam axis is parallel to $\mathbf{A}_3$, its length is L and the generic cross-section lies on a plane parallel to the plane spanned by $\{\mathbf{A}_1, \mathbf{A}_2\}$ and occupies the bounded region $D \subset \mathbb{R}^2$ with regular boundary ∂D. Consequently $V = D \times [0, L]$, with $[0, L] \subset \mathbb{R}$, and the position vector of the material point $P \equiv (x_1, x_2, \zeta)$ is furnished by

$$\mathbf{P}(\mathbf{x}, \zeta) = \mathbf{x} + \zeta \mathbf{A}_3 = x_\gamma \mathbf{A}_\gamma + \zeta \mathbf{A}_3, \quad (\mathbf{x}, \zeta) \in D \times [0, L] = V \tag{1}$$

(hereinafter repeated indexes denote summation, lower-case greek indexes assume the values 1,2, lower-case latin indexes assume the values 1,2,3 and capital latin indexes assume the values 0,1,2,3). The reference system is placed so that the positions of the cross section

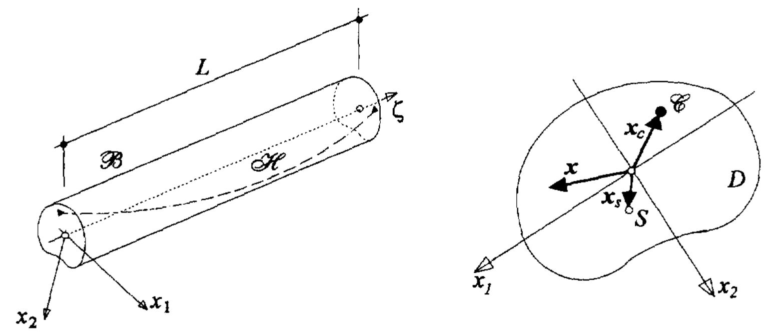

Fig. 1. Beam-cable system.

centroids have components $(0, 0, \zeta)$, while the shear centres S are located at $\mathbf{x}_s + \zeta\mathbf{A}_3$. Finally, A, I_1, I_2 and I_{12}, respectively, denote area and products of inertia of the cross section:

$$A = \int_D \mathrm{d}D, \quad I_1 = \int_D x_2^2\,\mathrm{d}D, \quad I_2 = \int_D x_1^2\,\mathrm{d}D, \quad I_{12} = I_{21} = \int_D x_1 x_2\,\mathrm{d}D. \tag{2}$$

The cable can slip at the interior of a tube rigidly linked to the beam. Under a geometric point of view, cable and tube have cross-sections which are extremely small with respect to their length, the beam length and the beam transversal dimension, so that it is possible and convenient to model them as a uni-dimensional manifold.

In particular, the tube $\mathscr{H}$ is modelled by a uni-dimensional manifold coinciding with its physical axis and is a part of the beam, i.e. $\mathscr{H} \subset \mathscr{B}$. In the reference configuration it traces the following regular curve:

$$\mathbf{H}(\zeta) = \mathbf{x}_c(\zeta) + \zeta\mathbf{A}_3 = x_{c\gamma}(\zeta)\mathbf{A}_\gamma + \zeta\mathbf{A}_3, \quad \zeta \in [0, L]; \tag{3}$$

by using primes for denoting the total derivative with respect to ζ, the tangent unit vector to the curve assumes the expression

$$\mathbf{G} = \frac{\mathbf{H}'}{|\mathbf{H}'|} = \frac{x'_{c\gamma}\mathbf{A}_\gamma + \mathbf{A}_3}{\sqrt{x'_{c\gamma}x'_{c\gamma} + 1}}, \tag{4}$$

while the total length on this reference configuration is furnished by the scalar quantity

$$\Lambda = \int_0^L |\mathbf{H}'|\,\mathrm{d}\zeta = \int_0^L \sqrt{x'_{c\gamma}x'_{c\gamma} + 1}\,\mathrm{d}\zeta. \tag{5}$$

Finally, the cable is modelled as a uni-dimensional manifold $\mathscr{C}$ whose points are identified by material co-ordinate $\rho \in [0, L]$. Let the initial cable position be described by the curve $\mathbf{R}(\rho)$ such that $\mathbf{R}(\rho) = \mathbf{H}(\zeta)$ for $\rho = \zeta$; in other terms it is supposed that the two parametrizations coincide in the reference configuration. The last assumption is used for convenience and permits intuitively defining the cable-beam slip on the basis of restrained kinematics for the cable. It is however important to observe that $\mathbf{H}(\rho)$ describes the position of the beam points while $\mathbf{R}(\rho)$ describes the position of the cable points.

2.2. *Deformation*

Rod and cable are two distinct geometrical entities, so that the generic system configuration is described by two different deformation functions. Description of kinematics is furnished according to the linear theory of the deformation.

With reference to the beam, under the fundamental assumption that the cross section is rigid in its own plane, a model sufficiently general to allow an accurate description of thin walled beam behaviour is adopted. More precisely, integration of the compatibility equations under the assumption of strain components null on the cross section plane leads to the following expression for the admissible displacement field (Davì, 1995):

$$\mathbf{p}(\mathbf{x}, \zeta) = \mathbf{P}(\mathbf{x}, \zeta) + \left[\mathbf{u}_0(\zeta) + \boldsymbol{\varphi}(\zeta) \times \mathbf{x} - \mathbf{I}_3\boldsymbol{\varphi}(\zeta) \times \mathbf{x}_s + \sum_{j=0}^{\infty} \chi_j(\zeta)\psi_j(\mathbf{x})\mathbf{A}_3\right], \tag{6}$$

in which $\mathbf{I}_3 = \mathbf{A}_3 \otimes \mathbf{A}_3$. The function $\mathbf{u}_0 = u_{0i}\mathbf{A}_i$ describes the beam axis displacements; $\boldsymbol{\varphi} = \varphi_i\mathbf{A}_i$ is the axial vector of the skew symmetric tensor $\mathbf{\Phi}(\zeta) = (\boldsymbol{\varphi} \times \mathbf{A}_i) \otimes \mathbf{A}_i$ (i.e. $\mathbf{\Phi}\mathbf{x} = \boldsymbol{\varphi} \times \mathbf{x}$) which describes the infinitesimal rigid rotation around a baricentric axis; χ_j are multiplicative coefficients of functions ψ_j defined on the cross section. It is assumed that the sequence constituted by ψ_j, by the constant term $\mathbf{u}_0 \cdot \mathbf{A}_3$ and by the linear term $\boldsymbol{\varphi} \times \mathbf{x} \cdot \mathbf{A}_3$ is complete in the considered space of the scalar functions defined on the cross section. The simpler beam models of Kirchhoff (Dill, 1992), Timoshenko (1945) and Vlasov (1961) may be reviewed as particular cases of that used in the present theory. The problem formulation is developed by considering the first 4 terms of the sequence with the aim of utilising, in the sequel and in the numerical applications, the warping functions ψ_j which are considered to be more meaningful in literature describing the behaviour of thin walled beams (Capurso 1984, Laudiero and Savoia 1990). Both warping functions and intensities of warping functions will be collected in the four component vectors $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$, respectively.

The deformed configuration is identified by prescribing the three vectorial functions $(\mathbf{u}_0, \boldsymbol{\varphi}, \boldsymbol{\chi})$ defined for $\zeta \in [0, L]$ which take values in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4$ and

$$C = \{\mathbf{v} \equiv (\mathbf{u}_0, \boldsymbol{\varphi}, \boldsymbol{\chi}) : [0, L] \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^4\}, \tag{7}$$

is the functional space of the beam deformations.

From (6), gradient $\mathbf{F}$ of the deformation field and gradient $\mathbf{U}$ of the displacement field $\mathbf{u} = \mathbf{p} - \mathbf{P}$ may be deduced in the following form:

$$\mathbf{F}(\mathbf{x}, \zeta) = \mathbf{I} + \mathbf{U}(\mathbf{x}, \zeta) = \mathbf{I} + \mathbf{f}_0(\mathbf{x}, \zeta) \otimes \mathbf{A}_3 + \mathbf{A}_3 \otimes \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}) + \boldsymbol{\Phi}, \tag{8}$$

where $\nabla_D(\cdot) = (\cdot)_{,\gamma}\mathbf{A}_\gamma$ denotes the gradient of scalar functions defined on D, while

$$\mathbf{f}_0(\mathbf{x}, \zeta) = \boldsymbol{\Gamma}(\zeta) + \boldsymbol{\varphi}' \times \mathbf{x} - \mathbf{I}_3 \boldsymbol{\varphi}' \times \mathbf{x}_s + (\boldsymbol{\chi}' \cdot \boldsymbol{\psi})\mathbf{A}_3, \tag{9a}$$

$$\boldsymbol{\Gamma}(\zeta) = \mathbf{u}_0' - \boldsymbol{\varphi} \times \mathbf{A}_3, \tag{9b}$$

Consequently, the linear strain tensor results

$$\mathbf{E}(\mathbf{x}, \zeta) = (\mathbf{f}_0 \otimes \mathbf{A}_3 + \mathbf{A}_3 \otimes \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}))^{\mathrm{Sym}} \tag{10}$$

(superscript Sym indicates the symmetric part of tensors) which, coherently with the assumption of transversally rigid cross section, does not have strain components in the cross section plane, i.e. $\mathbf{E} \cdot (\mathbf{I} - \mathbf{A}_3 \otimes \mathbf{A}_3) = 0$. The beam deformation is completely described by $\mathbf{d}(\zeta) \equiv (\boldsymbol{\Gamma}, \boldsymbol{\varphi}', \boldsymbol{\chi}, \boldsymbol{\chi}')$; in particular, the first two functions give the main shear strain which arise on the beam axis, the stretch of the beam axis itself and the axial strain due to the cross section rotations; the third gives the shear strain due to warping of the cross section and the last describes the contribution to longitudinal strain due to non-uniform warping along the beam axis.

Since the path $\mathscr{H}$ along which the cable can slip is rigidly linked to the beam, the actual deformation $\mathbf{p}$ may be deduced by means of the expression

$$\mathbf{h}(\zeta) = (\mathbf{p} \circ \mathbf{H})(\zeta) = \mathbf{p}^H(\zeta) = \mathbf{H} + \mathbf{u}_0 + \boldsymbol{\varphi} \times \mathbf{x}_c - \mathbf{I}_3 \boldsymbol{\varphi} \times \mathbf{x}_s + (\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H)\mathbf{A}_3 \tag{11}$$

where apex H indicates the composition $\circ\mathbf{H}$ describing the trace along the cable path $\mathscr{H}$ of a generic function defined on the beam points. Tangent vector $\mathbf{H}'$ transforms into

$$\begin{aligned}\mathbf{h}'(\zeta) &= (\mathbf{F} \circ \mathbf{H})(\zeta)\mathbf{H}'(\zeta) = \mathbf{F}^H\mathbf{H}' \\ &= \mathbf{H}' + \mathbf{u}_0' + \boldsymbol{\varphi}' \times \mathbf{x}_c + \boldsymbol{\varphi} \times \mathbf{x}_c' - \mathbf{I}_3 \boldsymbol{\varphi}' \times \mathbf{x}_s + [(\boldsymbol{\chi}' \cdot \boldsymbol{\psi}^H) + \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H) \cdot \mathbf{H}']\mathbf{A}_3\end{aligned} \tag{12}$$

which completely describes the deformation of an unitary segment of $\mathscr{H}$.

Regarding the cable, it is assumed that its ends are anchored at the ends of $\mathscr{H}$ and the cable traces, during each possible deformation $\mathbf{r}$, the same curve $\mathbf{p}^H(\zeta)$ traced by $\mathscr{H}$ even if slips may occur between the cable and the tube. This may be interpreted as a global constraint (Antman and Marlow, 1991) which reduces the set of generic deformations $\mathbf{r}$ of the cable to the subset of deformations for which $\mathbf{r}$ and $\mathbf{h}$ cover the same curve. The admissible deformations of the cable are related to the body deformation and can be represented by the relation (Dall'Asta and Leoni, 1995)

$$\mathbf{r}(\rho) = (\mathbf{p}^H \circ \alpha)(\rho), \tag{13}$$

where $\alpha(\rho) : [0, L] \to [0, L]$ is a bijective regular map related to the slips of the cable points along the path. In conclusion, to individuate the cable deformation it is not necessary to assign the vectorial function $\mathbf{r}$ but it is sufficient to know the scalar function α of the admissible relative slips. In other terms, kinematic descriptors which completely identify the system configuration are represented by the deformation function $\mathbf{p}$ and by the slip function α. The pure cable strain results

$$a(\rho) = \frac{|\mathbf{r}_{,\rho}|}{|\mathbf{R}_{,\rho}|} = \frac{|\mathbf{h}'(\alpha)|}{|\mathbf{R}_{,\rho}|}\alpha_{,\rho}. \tag{14}$$

In this work the case of homogeneous cable, free from external forces and in absence of friction is analysed, so that the cable internal stress is the same for each point of the cable as a consequence of the absence of external force and friction, while the stretch is constant thanks to the material homogeneity. In this particular case the introduction of the function α may be avoided, and the expression of a may be straight derived by utilising quantities related to the beam only, by evaluating the ratio between the actual length λ of the path $\mathscr{H}$ and its initial length Λ. Furthermore, even the function α and the whole cable deformation may be deduced from the beam deformation and from the cable path (Dall'Asta, 1995). The deformation $\mathbf{p}$ becomes the unique descriptor of the system kinematics and the pure deformation is furnished by the following expression

$$\begin{aligned} a &= \frac{\lambda}{\Lambda} = \frac{1}{\Lambda}\int_0^L |\mathbf{h}'|\,\mathrm{d}\zeta \\ &= \frac{1}{\Lambda}\int_0^L |\mathbf{H}' + \mathbf{u}_0' + \boldsymbol{\varphi}' \times \mathbf{x}_c + \boldsymbol{\varphi} \times \mathbf{x}_c' - \mathbf{I}_3\boldsymbol{\varphi}' \times \mathbf{x}_s + [(\boldsymbol{\chi}' \cdot \boldsymbol{\psi}^H) + \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H) \cdot \mathbf{H}']\mathbf{A}_3\,|\,\mathrm{d}\zeta. \end{aligned} \tag{15}$$

In such an expression, the functional dependence of a on $\mathbf{p}$, which expresses the coupling between the cable local deformation and the beam global deformations is evident. By linearizing the integral of module $\mathbf{h}'$ with respect to the displacements, the following expression can be obtained

$$a \cong 1 + \frac{1}{\Lambda}\int_0^L \frac{\mathbf{H}' \otimes \mathbf{H}'}{|\mathbf{H}'|} \cdot \mathbf{U}^H\,\mathrm{d}\zeta = 1 + \frac{1}{\Lambda}\int_0^L \frac{\mathbf{H}' \otimes \mathbf{H}'}{|\mathbf{H}'|} \cdot (\mathbf{f}_0^H \otimes \mathbf{A}_3 + \mathbf{A}_3 \otimes \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H))\,\mathrm{d}\zeta \tag{16}$$

from which it follows that, in the case of infinitesimal displacements, cable stretching is provided by the strain component along the tangent of the path while the antisymmetric part of $\mathbf{U}$ does not produce effects. For the particular beam model considered, a is a functional of the vectorial field $\mathbf{d}$ previously introduced.

Evaluation of a is necessary to determine the cable stress state. In any case, its deformation $\mathbf{r}$ may be deduced from $\mathbf{p}$ and the path geometry $\mathbf{H}$, as described in (Dall'Asta, 1996), while slips between the points of the cable and those of the protective tube can be easily calculated as difference between the length of the tube part and that of the cable part defined by the same parameter ζ

$$\delta = \int_0^{\zeta} \frac{\mathbf{H}' \otimes \mathbf{H}'}{|\mathbf{H}'|} \cdot \mathbf{U}^H \, d\xi - a \int_0^{\zeta} |\mathbf{H}'| \, d\xi. \tag{17}$$

2.3. *Thin walled beams: warping functions*

Up to now, no particular hypotheses on the cross section of the cylindrical solid are made so that the kinematic description of the previous section is valid for a generic beam. Geometrically, classical beams are characterised by a double order of dimensions related to the length of the beam and to the mean dimension of the cross section while thin walled beams have three orders of dimensions: the length, the mean dimension of the cross section and the wall thickness. By defining a curvilinear abscissa s supported by the trace of the mean wall surface $\mathscr{S}$, it is convenient to introduce a local reference system identified by the two unit vectors tangent ($\mathbf{a}_1$) and perpendicular ($\mathbf{a}_2$) to such a curve. The couple ($\mathbf{a}_1, \mathbf{a}_2$) is considered such that it may be superimposed on ($\mathbf{A}_1, \mathbf{A}_2$) by a plane roto-translation (Fig. 2).

Introduction of the local system does not modify the previous tensorial equations but permits immediately introducing the further assumptions which generally are on the grounds of the thin walled beam theories. The first hypothesis, thanks to the fact that the strain state is particularly simple to describe, is that direction $\mathbf{a}_2$ is perpendicular to the external surfaces of the beam, apart from the trace of the mean wall surface on the cross section; this is equivalent to assuming that the wall thickness t is constant and allows obtaining acceptable results even if t varies slowly along the cross section profile. In practical cases, in which the cross section presents sudden variations in thickness or branching, the developed theory is assumed to be locally valid except for the zones around the singular points.

Under the assumption that the wall thicknesses are very small, the warping functions may be assumed to be constant in direction $\mathbf{a}_2$ and thus depending on the curvilinear abscissa s only; it follows that

$$\psi(s,n) \cong \psi(s,0), \quad \nabla_D \psi_J \cong \psi_{J,s} \mathbf{a}_1. \tag{18}$$

It is assumed that the cross section warping be described by four different functions, by adopting the model described in detail in (Laudiero and Savoia, 1990). In this case, ψ_0 is the uniform torsion warping and, for open thin walled beams $\psi_0(s,0) = -\omega(s)$ (Vlasov, 1961), for which, by setting opportunely the curvilinear abscissa origin, they hold the orthogonality conditions

$$\int_D \psi_0 \, dD = 0, \quad \int_D (x_1 - x_{s1}) \psi_0 \, dD = 0, \quad \int_D (x_2 - x_{s2}) \psi_0 \, dD = 0. \tag{19}$$

The other functions ψ_1 and ψ_2 describe warping due to a generic shear force and ψ_3 the warping induced by shear flows due to non-uniform torsion. These may be calculated under

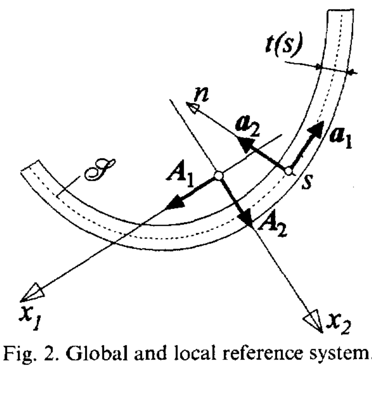

Fig. 2. Global and local reference system.

the assumption that they depend on the local beam resultants only and assume the form (Capurso 1984, Laudiero and Savoia, 1990)

$$\psi_j(s,0) = \bar{\psi}_j(s,0) - c_{1j}x_1(s) - c_{2j}x_2(s) - c_{\omega j}\omega(s) - c_j, \quad j = 1,2,3, \tag{20}$$

where $\bar{\psi}_j$ are calculated by integrating the

$$\frac{\mathrm{d}}{\mathrm{d}s}\bar{\psi}_1 = -\frac{S_2^*}{I_2 t}, \quad \frac{\mathrm{d}}{\mathrm{d}s}\bar{\psi}_2 = -\frac{S_1^*}{I_1 t}, \quad \frac{\mathrm{d}}{\mathrm{d}s}\bar{\psi}_3 = -\frac{S_\omega^*}{I_\omega t}, \tag{21}$$

in which

$$I_\omega = \int_D \omega^2 \,\mathrm{d}D, \quad S_1^* = \int_0^s x_2 t \,\mathrm{d}s, \quad S_2^* = \int_0^s x_1 t \,\mathrm{d}s, \quad S_\omega^* = \int_0^s \omega t \,\mathrm{d}s. \tag{22}$$

The twelve constants which appear in (20) may finally be calculated by imposing the orthogonality conditions

$$\int_D \psi_j \,\mathrm{d}D = 0, \quad \int_D x_1\psi_j \,\mathrm{d}D = 0, \quad \int_D x_2\psi_j \,\mathrm{d}D = 0, \quad \int_D \omega\psi_j \,\mathrm{d}D = 0, \quad j = 1,2,3. \tag{23}$$

3. BALANCE CONDITIONS FOR BEAM-CABLE SYSTEMS: STRESS AND EXTERNAL FORCES RESULTANTS

Balance conditions are obtained from the virtual work principle. Such a global approach is the most natural and simple for the problem examined because the cable strain has a functional dependence on the displacement functions defining the system state. Regarding the contribution to the internal virtual work related to the beam, the duality relations between dynamic and kinematic quantities described by **d**, and already known in literature, are obtained. On the other hand, internal work related to the cable allows determining new dynamic dual quantities for the cable. It can be observed that a formal analogy between the classical dynamic terms of the beam and the expression of the cable strain can be defined; this leads to a great simplification, under the formal aspect, of the balance condition. Finally, local equilibrium conditions for the beam and the cable are deduced from the global equilibrium condition given by the virtual work principle. This allows making explicit the resultants of the interaction forces which arise at the cable-beam interface and which, being of reactive nature, do not appear in the global balance condition. The analogy previously mentioned allows evidencing in a natural and synthetic way the relations which hold between the beam stress resultants and the cable stress.

The structural system is supposed to be subjected to external actions on the beam consisting of a contact force field $\mathbf{b}_S(\mathbf{x},\zeta)$ defined on the region ∂V_s of the beam surface, and of a force field $\mathbf{b}_V(\mathbf{x},\zeta)$ defined on the internal points. Let the stress beam state be described by the Cauchy tensor $\mathbf{T}(\mathbf{x},\zeta)$.

With regard to the cable, it is assumed that its internal stress state is described by the tension force $\tau\mathbf{G}(\zeta)$ tangent to the path and with constant intensity τ. Obviously anchorage and contact forces also act on the cable.

It is assumed that the virtual work principle for the system is expressed as

$$\int_V \mathbf{T}\cdot\hat{\mathbf{E}}\,\mathrm{d}V + \tau\hat{a}\Lambda = \int_V \mathbf{b}_V\cdot\hat{\mathbf{u}}\,\mathrm{d}V + \int_S \mathbf{b}_S\cdot\hat{\mathbf{u}}\,\mathrm{d}S \tag{24}$$

where $\hat{\mathbf{u}}$, $\hat{\mathbf{E}}$, $\hat{a}$ are the generic fields of admissible displacements and strains.

In the case under examination, admissible displacement variations are described by $\hat{\mathbf{v}} \equiv (\hat{\mathbf{u}}_0, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{\chi}}) \in C_0 \subset C$ where C_0 is the space of the displacements compatible with the external kinematic restraints. From variations of the admissible displacements, the quantities which describe the strain variation for the beam $\hat{\mathbf{d}} \equiv (\hat{\mathbf{\Gamma}}, \hat{\boldsymbol{\varphi}}', \hat{\boldsymbol{\chi}}, \hat{\boldsymbol{\chi}}')$ and for the cable $\hat{a}$ can be deduced.

Supposing that the admissible deformations of the beam are expressed by eqn (6), is equivalent to imposing on the beam a local constraint on the strain (Davì, 1995) while a global constraint is introduced when a finite number of warping functions is considered (Antman and Marlow, 1991). This reflects on the stress state which can be decomposed in two parts: the reactive part ($\mathbf{T}^R$) which is necessary to impose the constraint and which does not work by effect of admissible strains and the active part ($\mathbf{T}^A$) which derives from deformation of the body and which makes virtual work for admissible strains. In particular, introduction of the model leads to the following expression of the internal work related to the beam

$$\int_V \mathbf{T}\cdot\hat{\mathbf{E}}\,\mathrm{d}V = \int_V \mathbf{T}^A\cdot\hat{\mathbf{E}}\,\mathrm{d}V = \int_0^L [\mathbf{N}\cdot(\hat{\mathbf{u}}_0' - \hat{\boldsymbol{\varphi}}\times\mathbf{A}_3) + \mathbf{M}\cdot\hat{\boldsymbol{\varphi}}' + \boldsymbol{\upsilon}\cdot\hat{\boldsymbol{\chi}} + \boldsymbol{\mu}\cdot\hat{\boldsymbol{\chi}}']\,\mathrm{d}\zeta \tag{25}$$

and permits defining dynamic quantities dual of the kinematic descriptors. These quantities, generally known as resultant, moment, bi-shear and bi-moment, are obtained by integrating the active stresses on the cross section. Their explicit forms are the following ($\mathbf{t}_3^A(\mathbf{x};\zeta) = \mathbf{T}^A\mathbf{A}_3$):

$$\mathbf{N}(\zeta) = \int_D \mathbf{t}_3^A\,\mathrm{d}D,\quad \mathbf{M}(\zeta) = \int_D \mathbf{x}\times\mathbf{t}_3^A\,\mathrm{d}D - \mathbf{I}_3(\mathbf{x}_s\times\mathbf{N}),$$
$$\upsilon_I(\zeta) = \int_D \mathbf{t}_3^A\cdot\nabla_D\psi_I\,\mathrm{d}D,\quad \boldsymbol{\mu}(\zeta) = \int_D \boldsymbol{\psi} t_{33}^A\,\mathrm{d}D, \tag{26}$$

and can be collected in the vectorial function $\mathbf{Q}(\zeta)$ dual of $\mathbf{d}$ as follows

$$\mathbf{Q}(\zeta) = \begin{bmatrix}\mathbf{N}\\ \mathbf{M}\\ \boldsymbol{\upsilon}\\ \boldsymbol{\mu}\end{bmatrix}_{14\times 1} = \int_D \mathbf{B}(\mathbf{x},\zeta)\mathbf{t}_3^A(\mathbf{x},\zeta)\,\mathrm{d}D, \tag{27}$$

where the operator $\mathbf{B}(\mathbf{x};\zeta)$, which furnishes $\mathbf{Q}$ by integrating $\mathbf{B}\mathbf{t}_3$, is introduced.

With regard to the cable, the first step consists in evaluating the first variation of scalar $\hat{a}$ which results a linear functional of $\hat{\mathbf{d}}$ components with the following expression

$$\hat{a} = \frac{1}{\Lambda}\int_0^L \frac{\mathbf{H}'\otimes\mathbf{H}'}{|\mathbf{H}'|}\cdot\hat{\mathbf{E}}^H\,\mathrm{d}\zeta$$
$$= \frac{1}{\Lambda}\int_0^L |\hat{\mathbf{u}}_0' + \hat{\boldsymbol{\varphi}}'\times\mathbf{x}_c + \hat{\boldsymbol{\varphi}}\times\mathbf{x}_c' - \mathbf{I}_3\hat{\boldsymbol{\varphi}}'\times\mathbf{x}_s + [(\hat{\boldsymbol{\chi}}'\cdot\boldsymbol{\psi}^H) + \nabla_D(\hat{\boldsymbol{\chi}}\cdot\boldsymbol{\psi}^H)\cdot\mathbf{H}']\mathbf{A}_3|\,\mathrm{d}\zeta. \tag{28}$$

With some calculations it is easy to verify that the variation of the cable deformation may be expressed by means of the trace $\mathbf{B}^H$ of the previous operator $\mathbf{B}$ along the cable path $\mathbf{H}$, obtaining the form

$$\hat{a} = \frac{1}{\Lambda}\int_L \mathbf{B}^H(\zeta)\left[\frac{\mathbf{H}'(\zeta)\otimes\mathbf{H}'(\zeta)}{|\mathbf{H}'(\zeta)|}\mathbf{A}_3\right]\cdot\hat{\mathbf{d}}(\zeta)\,\mathrm{d}\zeta = \frac{1}{\Lambda}\int_L \mathbf{\Theta}(\zeta)\cdot\hat{\mathbf{d}}(\zeta)\,\mathrm{d}\zeta \tag{29}$$

in which $\mathbf{B}^H$ acts on the projection of $|\mathbf{H}'|\mathbf{A}_3$ along the curve tangent, by defining the vectorial function $\mathbf{\Theta} = [\mathbf{\Theta}_N, \mathbf{\Theta}_M, \mathbf{\Theta}_v, \mathbf{\Theta}_\mu]$ whose components are expressed by

$$\mathbf{\Theta}_N(\zeta) = \frac{\mathbf{H}'\otimes\mathbf{H}'}{|\mathbf{H}'|}\mathbf{A}_3, \quad \mathbf{\Theta}_M(\zeta) = \mathbf{x}_c\times\mathbf{\Theta}_N - \mathbf{I}_3(\mathbf{x}_s\times\mathbf{\Theta}_N),$$
$$\mathbf{\Theta}_{vI}(\zeta) = \mathbf{\Theta}_N\cdot\nabla_D\psi_I^H, \quad \mathbf{\Theta}_\mu(\zeta) = \psi^H(\mathbf{\Theta}_N\cdot\mathbf{A}_3). \tag{30}$$

The possibility of obtaining the resultants of stress fields on the beam and the pure deformation of the cable using the same operator derives from the problem geometry and from the deformation model. With regard to the beam, $\mathbf{B}$ acts on the traction $\mathbf{t}_3^A$ and furnishes dynamic quantities by integrating on the cross section, while its trace $\mathbf{B}^H$ appears in a different context in the case of the cable because it acts on a deformation quantity and furnishes a kinematical quantity, the variation of the cable strain, by integrating on the cable length, so that $\mathbf{\Theta}$ is not dual of $\mathbf{d}$. In the case of the cable, the field $\tau\mathbf{\Theta}$, generated by the constant force τ by the action of the kinematic model on the curve $\mathscr{H}$, may be interpreted as dual of $\mathbf{d}$.

The virtual work of the external forces may be stated in the form

$$\int_V \mathbf{b}_V\cdot\hat{\mathbf{u}}\,\mathrm{d}V + \int_{\partial V_s}\mathbf{b}_S\cdot\hat{\mathbf{u}}\,\mathrm{d}\partial V_s$$
$$= \int_0^L[\mathbf{n}\cdot\hat{\mathbf{u}}_0 + \mathbf{m}\cdot\hat{\boldsymbol{\varphi}} + \boldsymbol{\beta}\cdot\hat{\boldsymbol{\chi}}]\,\mathrm{d}\zeta + \mathbf{N}_\vartheta\cdot\hat{\mathbf{u}}_{0\vartheta}|_{\vartheta=0,L} + \mathbf{M}_\vartheta\cdot\hat{\boldsymbol{\varphi}}_\vartheta|_{\vartheta=0,L} + \boldsymbol{\mu}_\vartheta\cdot\hat{\boldsymbol{\chi}}_\vartheta|_{\vartheta=0,L}, \tag{31}$$

where the kinematic model induces the definition of the following stress resultants on the internal cross sections

$$\mathbf{n}(\zeta) = \int_D \mathbf{b}_V\,\mathrm{d}D + \int_{\partial D}\mathbf{b}_S\,\mathrm{d}\partial D, \quad \mathbf{m}(\zeta) = \int_D \mathbf{x}\times\mathbf{b}_V\,\mathrm{d}D + \int_{\partial D}\mathbf{x}\times\mathbf{b}_S\,\mathrm{d}\partial D - \mathbf{I}_3\mathbf{x}_s\times\mathbf{n},$$
$$\boldsymbol{\beta}(\zeta) = \int_D \boldsymbol{\psi}\mathbf{b}_V\cdot\mathbf{A}_3\,\mathrm{d}D + \int_{\partial D}\boldsymbol{\psi}\mathbf{b}_S\cdot\mathbf{A}_3\,\mathrm{d}\partial D, \tag{32}$$

which may be collected in the vectorial function $\mathbf{q}(\zeta) = [\mathbf{n}, \mathbf{m}, \boldsymbol{\beta}]$. On the end cross sections one obtains

$$\mathbf{n}_\vartheta = \int_{D_\vartheta}\mathbf{b}_S\,\mathrm{d}D_\vartheta, \quad \mathbf{m}_\vartheta = \int_{D_\vartheta}\mathbf{x}\times\mathbf{b}_S\,\mathrm{d}D_\vartheta - \mathbf{I}_3\mathbf{x}_s\times\mathbf{n}_\vartheta, \quad \boldsymbol{\beta}_\vartheta = \int_{D_\vartheta}\boldsymbol{\psi}(\mathbf{b}_S\cdot\mathbf{A}_3)\,\mathrm{d}D_\vartheta, \quad \vartheta = 0, L \tag{33}$$

which may be collected in the vectors $\mathbf{q}_\vartheta = [\mathbf{n}_\vartheta, \mathbf{m}_\vartheta, \boldsymbol{\beta}_\vartheta]$. Index $\vartheta = 0$, L denotes that the considered quantities are evaluated on the end basis.

To conclude, with the considered model, the balance condition can be stated in the form

$$\int_L \mathbf{Q}\cdot\hat{\mathbf{d}}\,\mathrm{d}\zeta+\tau\int_L \mathbf{\Theta}\cdot\hat{\mathbf{d}}\,\mathrm{d}\zeta=\int_L \mathbf{q}\cdot\hat{\mathbf{v}}\,\mathrm{d}\zeta+\mathbf{q}_\vartheta\cdot\hat{\mathbf{v}}_\vartheta|_{\vartheta=0,L}\quad \forall\hat{\mathbf{v}}\in\mathbf{C}_0. \tag{34}$$

It should be observed that in eqn (34) cable-beam interaction terms do not appear because, under the assumption of absence of friction, such forces are orthogonal to the path and are reactive. To obtain local information on the equilibrium of the beam, the integrals of eqn (34) may be integrated by parts and thus, thanks to the fundamental theorem of variational calculus, reduced to the following differential system

$$\mathbf{N}'=-(\mathbf{n}+\tau\mathbf{\Theta}'_N), \tag{35a}$$

$$\mathbf{M}'+\mathbf{A}_3\times\mathbf{N}=-\mathbf{m}-\tau(\mathbf{\Theta}'_M+\mathbf{A}_3\times\mathbf{\Theta}_N), \tag{35b}$$

$$\boldsymbol{\upsilon}-\boldsymbol{\mu}'=\boldsymbol{\beta}-\tau(\mathbf{\Theta}_\upsilon-\mathbf{\Theta}'_\mu), \tag{35c}$$

with the relevant boundary conditions

$$(\mathbf{N}+\tau\mathbf{\Theta}_N-\mathbf{n})_L\cdot\hat{\mathbf{u}}_{0L}=0\quad(\mathbf{N}+\tau\mathbf{\Theta}_N+\mathbf{n})_0\cdot\hat{\mathbf{u}}_0=0 \tag{36a}$$

$$(\mathbf{M}+\tau\mathbf{\Theta}_M-\mathbf{m})_L\cdot\hat{\boldsymbol{\varphi}}_L=0\quad(\mathbf{M}+\tau\mathbf{\Theta}_M-\mathbf{m})_0\cdot\hat{\boldsymbol{\varphi}}_0=0\quad\forall\hat{\mathbf{v}}_\vartheta \tag{36b}$$

$$(\boldsymbol{\mu}+\tau\mathbf{\Theta}_\mu-\boldsymbol{\beta})_L\cdot\hat{\boldsymbol{\chi}}_L=0\quad(\boldsymbol{\mu}+\tau\mathbf{\Theta}_\mu-\boldsymbol{\beta})_0\cdot\hat{\boldsymbol{\chi}}_0=0. \tag{36c}$$

These are local balance conditions for the beam and express the rule of the force field $\tau\mathbf{\Theta}$ per unit length generated by the cable stress. Such a field $\tau\mathbf{\Theta}$ describes interaction arising between beam and cables along the path $\mathscr{H}$ and may be interpreted as equivalent distributed actions, described by the vector $\bar{\mathbf{q}}$ with components

$$\bar{\mathbf{n}}=\tau\mathbf{\Theta}'_N,\quad \bar{\mathbf{m}}=\tau(\mathbf{\Theta}'_M+\mathbf{A}_3\times\mathbf{\Theta}_N),\quad \bar{\boldsymbol{\beta}}=-\tau(\mathbf{\Theta}_\upsilon-\mathbf{\Theta}'_\mu), \tag{37}$$

and as concentrated actions at the beam ends, described by vectors $\bar{\mathbf{q}}_\vartheta=-\tau\mathbf{\Theta}_\vartheta$.

In the particular case in which the τ value is known, as can occur during the stretching stages of cables, equivalent loads defined by relations (37) may be treated as external loads, since $\mathbf{\Theta}$ is constituted by functions deducible from the cable path only. In this case, the evaluation of the stress and strain states of the beam can be carried out separately from the cable. Obviously, the stress value is no longer known when external actions arise once the cable ends are fixed at the anchorages and relations (37) may be useful only later, once the deformations are known, to calculate the effective resultants of the force acting on the beam cross section.

4. THE ELASTIC PROBLEM

4.1. *Balance conditions*

Equations which govern the problem are determined by assuming the displacement field $\mathbf{v}(\zeta)$ as unknown and by introducing material constitutive laws. In particular it is assumed that the beam is constituted by a linear elastic material with the maximal material symmetry compatible with the constraint and the cable exhibits a linear elastic behaviour. The configuration in which the beam is in its natural state is assumed as the reference configuration. The equilibrium conditions are presented both in variational and local form. The coupling between global deformation of the beam and local strain of the cable leads to local equilibrium equations which are of integro-differential type.

The beam is homogeneous and constituted by a transversally isotropic material with the symmetry axis parallel to the beam axis. The active stress is consequently described by the Young's moduli E and G with the following relation:

$$\mathbf{T}^A = 2G\mathbf{E} + (E-2G)(\mathbf{E}\cdot\mathbf{I}_3)\mathbf{I}_3, \tag{38}$$

where $\mathbf{E}$ is the tensor of the admissible strains (eqn 10). From (38) the expression of the active stress state for the cylindrical solid is derived

$$\mathbf{T}^A = 2G(\mathbf{f}_0 \otimes \mathbf{A}_3 + \mathbf{A}_3 \otimes \nabla_D(\boldsymbol{\chi}\cdot\boldsymbol{\psi}))^{\mathrm{Sym}} + (E-2G)f_{03}\mathbf{I}_3. \tag{39}$$

So that the active traction on the cross section $\mathbf{t}_3^A = \mathbf{T}^A\mathbf{A}_3$ is

$$\mathbf{t}_3^A = G[u'_{0\gamma}\mathbf{A}_\gamma - \varphi_\gamma\mathbf{A}_\gamma \times \mathbf{A}_3 + \varphi'_3\mathbf{A}_3 \times (\mathbf{x}-\mathbf{x}_s) + \nabla_D(\boldsymbol{\chi}\cdot\boldsymbol{\psi})] + E[u'_{03}\mathbf{A}_3 + \varphi'_\gamma\mathbf{A}_\gamma \times \mathbf{x} + (\boldsymbol{\chi}'\cdot\boldsymbol{\psi})\mathbf{A}_3]. \tag{40}$$

If normality conditions (19, 23) hold and the reference frame coincides with the inertia principal reference frame, this expression of the traction fields on the section permits defining the constitutive relation between the cross section stress resultants (26) and the components of the strain descriptors $\mathbf{d}$ in the form

$$\mathbf{N} = GA(u'_{0\gamma}\mathbf{A}_\gamma - \varphi_\gamma\mathbf{A}_\gamma \times \mathbf{A}_3) - GA(\varphi'_3\mathbf{A}_3 \times \mathbf{x}_s) + GC_{J\gamma}\chi_J\mathbf{A}_\gamma + EAu'_{03}\mathbf{A}_3, \tag{41a}$$

$$\begin{aligned}\mathbf{M} = {} & G[J - C_{s0}]\varphi'_3\mathbf{A}_3 + G\chi_J C_{sJ}\mathbf{A}_3 \\ & - GA[x_{s1}(u'_{02}+\varphi_1) - x_{s2}(u'_{01}-\varphi_2)]\mathbf{A}_3 + E(I_1\varphi'_1\mathbf{A}_1 + I_2\varphi'_2\mathbf{A}_2),\end{aligned} \tag{41b}$$

$$\mu_I = E\chi'_J I_{\psi_J\psi_I}, \tag{41c}$$

$$v_I = (u'_{01}-\varphi_2)GC_{I1} + (u'_{02}+\varphi_1)GC_{I2} + \varphi_3 GC_{sI} + G\chi_J C_{\psi_I\psi_J}, \tag{41d}$$

In the previous eqns (41), the following inertia characteristics were introduced

$$J = \int_D (\mathbf{x}\cdot\mathbf{x})\,\mathrm{d}D + A(\mathbf{x}_s\cdot\mathbf{x}_s) + C_{s0}, \tag{42a}$$

$$I_{\psi_I\psi_J} = \int_D \psi_I\psi_J\,\mathrm{d}D, \quad C_{\psi_I\psi_J} = \int_D (\nabla_D\psi_I\cdot\nabla_D\psi_J)\,\mathrm{d}D, \tag{42b,c}$$

$$C_{J_\gamma} = \int_D \nabla_D\psi_J\cdot\mathbf{A}_\gamma\,\mathrm{d}D, \quad C_{sJ} = \int_D |(\mathbf{x}-\mathbf{x}_s)\times\nabla\psi_J|\,\mathrm{d}D. \tag{42d,e}$$

The first is the DSV torsional moment of inertia while the others are related to cross section warping. In particular, indexes I and J equal to 0 furnish the inertia defined in the Vlasov non uniform torsion theory. Relations (41) represent the constitutive law for the beam element. In synthetic form it holds

$$\mathbf{Q}(\zeta) = \mathbf{K}\mathscr{D}\mathbf{v}(\zeta), \quad \zeta \in [0, L], \tag{43}$$

where $\mathbf{Q}$ is the generalised force vector which collects the internal stress resultants, $\mathbf{K}$ is the stiffness matrix for the general beam element and $\mathbf{v} \to \mathscr{D}\mathbf{v}$ is the formal linear differential operator which associates $\mathbf{d}$ to $\mathbf{v}$ and is defined as

$$\mathscr{D}\mathbf{v} = \mathbf{d} = \begin{bmatrix} \mathbf{u}_0' - \boldsymbol{\varphi} \times \mathbf{A}_3 \\ \boldsymbol{\varphi}' \\ \boldsymbol{\chi} \\ \boldsymbol{\chi}' \end{bmatrix}_{14 \times 1}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{\Gamma\Gamma} & \mathbf{K}_{\Gamma\varphi'} & \mathbf{K}_{\Gamma\chi} & \mathbf{0} \\ \mathbf{K}_{\varphi'\Gamma} & \mathbf{K}_{\varphi'\varphi'} & \mathbf{K}_{\varphi'\chi} & \mathbf{0} \\ \mathbf{K}_{\chi\Gamma} & \mathbf{K}_{\chi\varphi'} & \mathbf{K}_{\chi\chi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{\chi'\chi'} \end{bmatrix}_{14 \times 14}. \tag{44}$$

With regard to the cable, under the assumption that it is homogeneous and constituted by a linear elastic material, stress is measured by $\tau = c(e+e_0)$, where c is the stiffness constant, $e = a-1$ is the cable strain developed from the reference configuration and e_0 is the strain measured in the reference configuration with respect to the natural configuration of the cable ($\tau = 0$). In this case, for eqn (39), the beam stress state is null in the reference configuration which is, however, not necessarily balanced in absence of external loads because the cable is prestressed. Denoting by c the stiffness of the cable and by ce_0 the traction force acting on it in the reference configuration, the constitutive law of the cable assumes the form

$$\tau = c(a(\mathbf{v})-1+e_0) = \frac{c}{\Lambda}\int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\mathbf{v}\, \mathrm{d}\zeta + ce_0, \tag{45}$$

in which the dependence on the displacements $\mathbf{v}$ is also made explicit. Since the cable is homogeneous τ is a constant.

As seen in the previous chapter, eqn (34) represents the global balance condition of the beam-cable system, on the other hand eqns (35) characterise the local equilibrium of the beam element. The solution of the elastic problem may be performed by using one of the two formulations by making explicit the dynamic quantities previously defined in terms of displacement functions.

Considering eqn (34) the weak form of the problem is represented by the equation

$$\int_0^L \mathbf{K}\mathscr{D}\mathbf{v} \cdot \mathscr{D}\hat{\mathbf{v}}\, \mathrm{d}\zeta + \frac{c}{\Lambda}\left(\int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\mathbf{v}\, \mathrm{d}\zeta\right)\left(\int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\hat{\mathbf{v}}\, \mathrm{d}\zeta\right)$$
$$= -ce_0 \int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\hat{\mathbf{v}}\, \mathrm{d}\zeta + \int_0^L \mathbf{q} \cdot \hat{\mathbf{v}}\, \mathrm{d}\zeta + \mathbf{q}_\vartheta \cdot \hat{\mathbf{v}}_\vartheta|_{\vartheta=0,L}. \tag{46}$$

The substantial difference between the beam term, in which the virtual work depends only on local quantities, and the term related to the cable, in which stress and strain have a functional dependence on the deformation, is reflected on the different formal structure of the two terms, the former described by an integral of a quantity quadratic on the displacements and the latter described by the product of two integrals which are linear in the displacements.

Obviously, the solution belongs to the space of the functions which fulfil the kinematic boundary conditions and which are sufficiently regular to give a sense to the quadratic and linear integrals of eqn (45) in the Lebesque integration theory. Under the assumption that the constitutive relations are continuous and limited, the following space of the unknowns is assumed

$$U = \left\{\mathbf{v} \in C;\quad \|\mathbf{v}\|_U = \left[\int_0^L |\mathscr{D}\mathbf{v}|^2 + |\mathbf{v}|^2\, \mathrm{d}\zeta\right]^{1/2} < \infty\right\}. \tag{47}$$

By considering (35) and (36), for (43) and (45), the strong formulation is constituted by the following differential system

$$\mathscr{D}^*\mathbf{K}\mathscr{D}\mathbf{v}(\zeta)+\mathbf{q}(\zeta)+c\left(\frac{1}{\Lambda}\int_0^L \mathbf{\Theta}\cdot\mathscr{D}\mathbf{v}\,d\zeta+e_0\right)\mathscr{D}^*\mathbf{\Theta}(\zeta) = \mathbf{0}, \quad \zeta\in(0,L) \tag{48}$$

with the relevant boundary conditions

$$\left[\mathscr{L}\mathbf{K}\mathscr{D}\mathbf{v}_L+c\left(\frac{1}{\Lambda}\int_0^L \mathbf{\Theta}\cdot\mathscr{D}\mathbf{v}\,d\zeta+e_0\right)\mathscr{L}\mathbf{\Theta}-\mathbf{q}_L\right]\cdot\hat{\mathbf{v}}_L = \mathbf{0}, \tag{49a}$$

$$\left[\mathscr{L}\mathbf{K}\mathscr{D}\mathbf{v}_0+c\left(\frac{1}{\Lambda}\int_0^L \mathbf{\Theta}\cdot\mathscr{D}\mathbf{v}\,d\zeta+e_0\right)\mathscr{L}\mathbf{\Theta}+\mathbf{Q}_0\right]\cdot\hat{\mathbf{v}}_0 = \mathbf{0}, \quad \forall\hat{\mathbf{v}}_9 \tag{49b}$$

where $\mathbf{Q}\rightarrow\mathscr{D}^*\mathbf{Q}$ and $\mathbf{Q}\rightarrow\mathscr{L}\mathbf{Q}$ are the operators defined as

$$\mathscr{D}^*\mathbf{Q} = \begin{bmatrix}\mathbf{N}'\\ \mathbf{M}'+\mathbf{A}_3\times\mathbf{N}\\ \upsilon-\boldsymbol{\mu}'\end{bmatrix}_{10\times 1}, \quad \mathscr{L}\mathbf{Q} = \begin{bmatrix}\mathbf{N}\\ \mathbf{M}\\ \boldsymbol{\mu}\end{bmatrix}_{10\times 1}. \tag{50}$$

The formulation obtained is not of a differential type but is an integro-differential type formulation. The equations obtained express the equilibrium of an infinitesimal portion of the system identified by two planes orthogonal to $\mathbf{A}_3$ and the integral on the displacements denote that the forces depend on the deformation of the whole beam. In this case the problem solution belongs to a space of function smaller to that previously defined, in which it is possible to define the derivatives.

The strong formulation of the problem was written for the sake of completeness but, as observed in (Dall'Asta and Dezi, 1993), the variational formulation of the problem is the most natural and permits obtaining approximate solutions with classical methods. Usually, the local formulation of the classical beam problem is very interesting under the qualitative aspect because it allows obtaining analytic solutions for simple problems but which are of practical application in engineering. This is no longer possible for the integro-differential system which describes the problem under examination.

4.2. *Remarks on thin walled beams*

Inertial characteristics of the cross section defined by eqns (42) are valid in general. If $\mathscr{S}$ indicates the trace of the wall mean surface of the beam, in the local reference frame as defined in Section 2.3, the integrals on the cross section may be rewritten in the form

$$\int_D f(\mathbf{x})\,dD = \int_{\mathscr{S}}\int_{-t/2}^{t/2} f(s,n)\,dn\,ds = \int_{\mathscr{S}} f(s,0)t+O(t^3)\,ds. \tag{51}$$

If t is much smaller than the length of the mean line of the cross section, powers of an order higher than or equal to 3 may be disregarded so that eqns (42b,c,d,e) reduce to

$$I_{\psi_I\psi_J} = \int_{\mathscr{S}} \psi_I(s,0)\psi_J(s,0)t\,ds, \quad C_{\psi_I\psi_J} = \int_{\mathscr{S}} \psi_{I,s}(s,0)\psi_{J,s}(s,0)t\,ds,$$

$$C_{J\gamma} = \int_{\mathscr{S}} \psi_{J,s}(s,0)t\,ds\mathbf{a}_1\cdot\mathbf{A}_\gamma, \quad C_{sJ} = \int_{\mathscr{S}} \psi_{J,s}(s,0)|[\mathbf{x}(s,0)-\mathbf{x}_s]\times\mathbf{a}_1|t\,ds. \tag{52}$$

Inertia $I_{\psi_I\psi_J}$ for indexes $I\neq J$ are much smaller than those obtained with $I=J$ so that in the sequel they will be disregarded (Laudiero *et al.*, 1991).

A particular discussion is required for eqn (42a) for which a reasoning similar to the previous does not hold in that it could lead to bad estimation of the real value of J. In the case of thin walled beam with open cross section, the gradient of the warping function due to pure torsion has two components which arc comparable and the hypothesis of constant warping on the wall thickness has to be removed. In such a case a good estimation of J is given by the well known

$$J = \frac{1}{3}\sum_{i=1}^{n_b} t_i^3 b_i. \tag{53}$$

Once the elastic problem is solved, or rather, once the equilibrated configuration of the beam-cable system has been determined, the active stress components, are known. Generally, such stresses are not sufficient to guarantee local equilibrium which also requires reactive stresses. For a generic cross section, the local balance conditions valid for three-dimensional bodies are not sufficient to calculate the reactive stresses so that they are indeterminate. However, an estimation of such stresses is necessary both to carry out the test on the punctual strength and to perform the stability analysis of the equilibrated configurations (Como and Grimaldi, 1995).

The total stress of a thin walled beam is usually performed by accepting that the active traction t_{33}^A is an acceptable estimation of the stress t_{33} and the transversal stress t_{ss} is constant in the thickness while the other transversal components t_{sn} and t_{n3} are null. The active reaction t_{33}^A is given by

$$\mathbf{t}_3^A \cdot \mathbf{A}_3 = t_{33}^A(s,\zeta) = \frac{N_3}{A} + \frac{M_1}{I_1}x_2 - \frac{M_2}{I_2}x_1 + \frac{\mu_J \psi_J}{I_{\psi_J\psi_J}}, \tag{54}$$

The shear component t_{3s}, related to directions $\mathbf{a}_1$–$\mathbf{A}_3$, can be deduced from the internal equilibrium condition $t_{3s,s} = -b_{V3} - t_{33,\zeta}$ taking into account the values assumed at the edges $t_{3s} = -b_{S3}$, it results

$$\mathbf{t}_3 \cdot \mathbf{a}_1 = t_{3s}(s,\zeta) = \frac{n_3 + \bar{n}_3}{At}s - \frac{(N_2 - m_1 - \bar{m}_1)S_1^*}{tI_1} - \frac{(N_1 + m_2 + \bar{m}_2)S_2^*}{tI_2} - \frac{(v_J - \beta_J - \bar{\beta}_J)S_{\psi J}^*}{tI_{\psi_J\psi_J}} - b_{s3}, \tag{55}$$

In the case of beams with cross section characterised by $I_{\psi_0\psi_0} = 0$, eqn (55) cannot furnish stress distributions which can equilibrate torsional moments so that it is necessary to superimpose, on the stress distribution uniform on the wall thickness, a triangular shear stress distribution given by

$$t_{3s}^{\mathrm{DSV}}(s,n,\zeta) = \frac{6M_3^{\mathrm{DSV}}}{\sum_{i=1}^{n_b} t_i^3 b_i} n. \tag{56}$$

where $M_3^{\mathrm{DSV}} = GJ\varphi_3'$ is the DSV contribution to the total twisting moment.

An estimation of the more important transversal stress component can be obtained from the equilibrium condition $t_{ss,s} = -b_{Vs} - t_{s3,\zeta}$ in the internal points and the equilibrium condition $t_{ss} = -b_{Ss}$ at the contour edges. The third balance condition in the direction of $\mathbf{a}_2$, is exactly satisfied only if the loads compatibility condition $b_{V_n} = -|\mathbf{a}_{1,s}| t_{ss}$ holds (Gjelsvick, 1981).

5. INFINITESIMAL STABILITY OF EQUILIBRIUM CONFIGURATIONS

5.1. *Stability conditions*

Balance conditions, adopted in the previous chapter, were deduced by linearizing the relations of the non linear theory for a reference configuration, not necessarily equilibrated, in which the beam is in its natural state, i.e., zero stress. This leads to a well conditioned problem and generally furnishes a realistic prediction of the stress and strain of the system, at least in structural analysis. In this chapter, by linearizing the problem in the neighbourhood of a given equilibrated configuration (Truesdell and Noll, 1965), the stability of the beam-cable system subjected to external force and to the cable prestressing will be studied. The reference configuration is chosen coincident with the known configuration. As previously, the admissible displacement fields are introduced in the balance conditions obtained in (Dall'Asta, 1996) for generic unconstrained three-dimensional bodies.

Stability of the system is related to the following bi-linear form

$$\Psi(\mathbf{v},\hat{\mathbf{v}}) = \int_V \mathbf{C}[\mathbf{E}]\cdot\hat{\mathbf{E}}\,\mathrm{d}V - \frac{c}{\Lambda}\left(\int_0^L \mathbf{G}\cdot\mathbf{U}^H\mathbf{H}'\,\mathrm{d}\zeta\right)\left(\int_0^L \mathbf{G}\cdot\hat{\mathbf{U}}^H\mathbf{H}'\,\mathrm{d}\zeta\right) + \int_V \mathbf{T}_0\cdot\mathbf{U}^T\hat{\mathbf{U}}\,\mathrm{d}V$$

$$+\tau_0\int_0^L \frac{\mathbf{I}-\mathbf{G}\otimes\mathbf{G}}{|\mathbf{H}'|}\cdot(\mathbf{U}^H\mathbf{H}'\otimes\hat{\mathbf{U}}^H\mathbf{H}')\,\mathrm{d}\zeta. \quad (58)$$

More precisely the system will be stable if such a bilinear form is positive defined, i.e. $\exists\alpha > 0 : \Psi(\mathbf{v},\mathbf{v}) \geqslant \alpha\|\mathbf{v}\|_U$.

The first and the third terms are the classical terms which appear in the stability analysis of three-dimensional bodies. The first is related to the constitutive quantity C; consequently it is positive defined if the material is stable everywhere for the stress field under examination, so that it furnishes a stabilising contribution to the system which, for the case of cylindrical bodies with admissible displacement field described by eqn (6), may be written as

$$\int_V \mathbf{C}[\mathbf{E}]\cdot\hat{\mathbf{E}}\,\mathrm{d}V = \int_0^L \mathbf{K}\mathscr{D}\mathbf{v}\cdot\mathscr{D}\hat{\mathbf{v}}\,\mathrm{d}\zeta, \quad (59)$$

where the differential operator $\mathscr{D}$ previously defined is utilised.

The third term of eqn (58) is the classical geometrical component which represents the internal work performed by the stress state by effect of the strain quadratic term induced by the variation of the equilibrated configuration. It is not defined in sign and may be responsible for eventual system instability. It is important to observe that the instabilising effects depend on the whole stress tensor and so derive both from its active and reactive parts. In particular, in the case of thin walled beam, the reactive stresses may play a fundamental role in the evaluation of the critical load, as in the case of simply supported beam with load resultant not applied at the shear centre. By making explicit the dependence of the displacement field on quantities describing the beam deformation, one obtains

$$\int_V \mathbf{T}\cdot\mathbf{U}^T\hat{\mathbf{U}}\,\mathrm{d}V = \int_V \mathbf{T}\cdot[(\mathbf{f}_0\cdot\hat{\mathbf{f}}_0)\mathbf{A}_3\otimes\mathbf{A}_3]\,\mathrm{d}V$$

$$+\int_V \mathbf{T}\cdot[(f_{03}\mathbf{I}-\mathbf{\Phi})(\mathbf{A}_3\otimes\nabla_D(\hat{\chi}\cdot\psi))+(\nabla_D(\chi\cdot\psi)\otimes\mathbf{A}_3)(\hat{f}_{03}\mathbf{I}+\hat{\mathbf{\Phi}})]\,\mathrm{d}V$$

$$+\int_V \mathbf{T}\cdot[\nabla_D(\psi\cdot\chi)\otimes\nabla_D(\psi\cdot\hat{\chi})+[(\mathbf{A}_3\otimes\mathbf{f}_0)\hat{\mathbf{\Phi}}-\mathbf{\Phi}(\hat{\mathbf{f}}_0\otimes\mathbf{A}_3)]-\mathbf{\Phi}\hat{\mathbf{\Phi}}]\,\mathrm{d}V. \quad (60)$$

By introducing the coefficient matrix **W** related to the stress components and to the inertial quantities of the cross section, and the linear differential formal operator $\mathbf{v} \rightarrow \mathscr{G}\mathbf{v}$ defined as follows,

$$\mathscr{G}\mathbf{v} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_0' \\ \boldsymbol{\varphi} \\ \boldsymbol{\varphi}' \\ \boldsymbol{\chi} \\ \boldsymbol{\chi}' \end{bmatrix}_{20\times 1}, \tag{61}$$

the previous equation may be rewritten in the synthetic form

$$\int_V \mathbf{T}_0 \cdot \mathbf{U}^T \hat{\mathbf{U}} \, \mathrm{d}V = \int_0^L \mathbf{W}\mathscr{G}\mathbf{v} \cdot \mathscr{G}\hat{\mathbf{v}} \, \mathrm{d}\zeta. \tag{62}$$

The remaining second and fourth terms of eqn (58) involve the cable. They depend on the quantity $\mathbf{U}^H\mathbf{H}'$ which expresses the difference between the unit element of the path in the deformed configuration and that in the undeformed configuration. Such a quantity may be deduced from the functions describing the beam deformation by means of the relation

$$\mathbf{U}^H\mathbf{H}' = \mathbf{h}' = \mathbf{u}_0' + (\varphi_\gamma \mathbf{A}_\gamma \times \mathbf{x}_c)' + [\varphi_3 \mathbf{A}_3 \times (\mathbf{x}_c - \mathbf{x}_s)]' + (\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H)' \mathbf{A}_3. \tag{63}$$

A term with constitutive nature and a term related to the stress state may also be recognised for the cable. Even if c is a constitutive quantity for the cable and thus positive defined in the range of stable behaviour of the material, the first is only positive semi-defined. It gives a stabilising contribution only for those displacements fields which have a non null trace along path $\mathscr{H}$. Furthermore, $\mathbf{G}$ and $\mathbf{H}'$ are parallel so that only the pure strain $\mathbf{E}^H$ of the displacement gradient $\mathbf{U}^H$ has a stabilising effect. By using the operators previously described, a second term of eqn (58) may be written in the form.

$$\frac{c}{\Lambda}\left(\int_0^L \mathbf{G} \cdot \mathbf{U}^H \mathbf{H}' \, \mathrm{d}\zeta\right)\left(\int_0^L \mathbf{G} \cdot \hat{\mathbf{U}}^H \mathbf{H}' \, \mathrm{d}\zeta\right) = \frac{c}{\Lambda}\left(\int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\mathbf{v} \, \mathrm{d}\zeta\right)\left(\int_0^L \boldsymbol{\Theta} \cdot \mathscr{D}\hat{\mathbf{v}} \, \mathrm{d}\zeta\right). \tag{64}$$

Finally, the last term of the bilinear form, related to the internal force on the cable, is analysed. It should be observed that the cable is stretched and designed to maintain a positive stress even under the action of the external loads, furthermore the integrated function is non negative for $\hat{\mathbf{v}} = \mathbf{v}$. It follows that this term may produce a stabilising effect but, unlike the correspondent beam term, it is only positive semi-defined because the integral is null for the fields of $\mathbf{U}$ which have a trace null on $\mathbf{H}$. It can be deduced that a substantial difference exists, in stability, between a beam characterised by a stress field $\mathbf{T}_0$ produced by external loads and a beam in which the same stress state is produced by an internal stretched cable. In fact, if the first may be in an unstable configuration, the second may still be stable. This occurs every time the deformation actuated in the first case involves displacements which induce transversal deformation of the cable path. To make this last term explicit as function of the deformation model assumed, one obtains

$$\int_0^L \frac{\mathbf{I}-\mathbf{G}\otimes\mathbf{G}}{|\mathbf{H}'|}\cdot(\mathbf{U}^H\mathbf{H}'\otimes\hat{\mathbf{U}}^H\mathbf{H}')\,\mathrm{d}\zeta$$

$$= \int_0^L \frac{\mathbf{I}-\mathbf{G}\otimes\mathbf{G}}{|\mathbf{H}'|}\cdot\{[u'_{0\gamma}\mathbf{A}_\gamma+(\varphi_3\mathbf{A}_3\times(\mathbf{x}_c-\mathbf{x}_s))']\otimes[\hat{u}'_{0\gamma}\mathbf{A}_\gamma+(\hat{\varphi}_3\mathbf{A}_3\times(\mathbf{x}_c-\mathbf{x}_s))']\}\,\mathrm{d}\zeta. \quad (65)$$

so that, by introducing the coefficient matrix $\mathbf{W}^c$ related only to the path geometry, the following equation may be deduced:

$$\int_0^L \frac{\mathbf{I}-\mathbf{G}\otimes\mathbf{G}}{|\mathbf{H}'|}\cdot(\mathbf{U}^H\mathbf{H}'\otimes\hat{\mathbf{U}}^H\mathbf{H}')\,\mathrm{d}\zeta = \int_0^L \mathbf{W}^c\mathscr{G}\mathbf{v}\cdot\mathscr{G}\hat{\mathbf{v}}\,\mathrm{d}\zeta. \quad (66)$$

Apart from the functional dependence of the cable deformation terms on the trace of the beam deformation terms, from eqn (58) it can be observed that components of $\mathbf{U}^H\mathbf{H}'$ along the tangent direction to the curve contribute only to the constitutive term, while the complementary components along the orthogonal direction to the path contribute only to the term related to force τ_0.

Generally, in structural mechanics problems, the stress state of the known configuration is obtained by a linear analysis, assuming that the solution may be reasonably described under the infinitesimal displacement and strain assumption and that the material works in its linear range. This situation permits subdividing the geometric term (62) in two parts, one due to the stress state induced by prestressing and another due to the stress state induced by external actions. So that the Cauchy tensor may be rewritten as

$$\mathbf{T}_0 = \tau_0\mathbf{T}_{0\tau}+\kappa\mathbf{T}_{0\kappa}, \quad (67)$$

in which κ is a multiplier of the external loads while $\mathbf{T}_{0\tau}$ and $\mathbf{T}_{0\kappa}$ represent stress induced by the cable unit tension and by the external load combination characterised by $\kappa = 1$, respectively.

A similar observation must be made on the cable stress τ_0 which, according to (45), may decompose as

$$\tau_0 = \tau_0^*+\kappa\tau_{0\kappa} \quad (68)$$

where τ_0^* is the initial cable stretching stress while $\tau_{0\kappa}$ is the stress increment due to external loads for $\kappa = 1$.

The bilinear form may be written in the form

$$\Psi(\mathbf{v},\hat{\mathbf{v}}) = \int_0^L \mathbf{K}\mathscr{D}\mathbf{v}\cdot\mathscr{D}\hat{\mathbf{v}}\,\mathrm{d}\zeta+\frac{c}{\Lambda}\left(\int_0^L \mathbf{\Theta}\cdot\mathscr{D}\mathbf{v}\,\mathrm{d}\zeta\right)\left(\int_0^L \mathbf{\Theta}\cdot\mathscr{D}\hat{\mathbf{v}}\,\mathrm{d}\zeta\right)$$

$$+\tau^*\int_0^L(\mathbf{W}_\tau+\mathbf{W}^c)\mathscr{G}\mathbf{v}\cdot\mathscr{G}\hat{\mathbf{v}}\,\mathrm{d}\zeta+\kappa\int_0^L[\tau_\kappa(\mathbf{W}_\tau+\mathbf{W}^c)+\mathbf{W}_\kappa]\mathscr{G}\mathbf{v}\cdot\mathscr{G}\hat{\mathbf{v}}\,\mathrm{d}\zeta, \quad (69)$$

in which matrixes $\mathbf{W}_\tau$ and $\mathbf{W}_\kappa$ derive from the decomposition of the Cauchy tensor (67). The previous hypothesis on the modality used to reach the equilibrated configuration to be analysed, permit separating the effects due to the cable stretching force from those due to external actions which are proportional to κ.

At this point it is natural to define κ_{cr} and τ_{0cr}^* as the multipliers of the loads and the initial stretching force which make the bilinear form *non positive* defined. In particular, by assuming the absence of external actions and focusing attention on the cable effects, it may be observed that if the following condition

$$\int_0^L (\mathbf{W}_\tau + \mathbf{W}^c)\mathscr{G}\mathbf{v} \cdot \mathscr{G}\hat{\mathbf{v}}\,d\zeta > 0 \quad \forall \hat{\mathbf{v}} \in U, \tag{70}$$

is verified it may be concluded that the bilinear form (69) is positive defined for each initial cable force and, consequently, τ^*_{0cr} does not exist. On the other hand, if admissible displacement fields $\hat{\mathbf{v}}^*$, such that

$$\int_0^L (\mathbf{W}_\tau + \mathbf{W}^c)\mathscr{G}\hat{\mathbf{v}}^* \cdot \mathscr{G}\hat{\mathbf{v}}^*\,d\zeta < 0, \tag{71}$$

exist it may be concluded that a sufficiently high τ^*_{0cr}, for which the whole bilinear form becomes non positive, exists. In any case, the amplitude of the corresponding stresses and of the stretching force must be tested to establish their compatibility with the basic assumption for the determination of the reference configuration.

The integral of inequality (70) holds both the bilinear form related to the cable with internal force $\tau^*_0 = 1$ and that due to the beam stress required by equilibrium in such a condition. After all, $\mathbf{W}_\tau$ and $\mathbf{W}^c$ are matrixes which may depend only on the path geometry $\mathbf{H}$ and on the beam geometry (shape of the cross section and length). Thus, once a beam is chosen, the definition of cable *stabilising path* (*instabilising*) may be introduced for those curves $\mathbf{H}$ which fulfil (*do not*) inequality (70), regardless of the stretching force value. It must be observed that the introduced definition also depends unavoidably on the displacement model adopted so that a stabilising path with respect to inequality (70) will still be such for poorer deformation models like those of Vlasov, Timoshenko or Kirchhoff, while it may no longer be such in the case of more refined models which contemplate U as subset, or simply in the case of different models. The observation is of practical interest especially in those cases in which it is necessary to analyse local buckling. Generally, in these cases, more refined displacement models, in which the cross section may deform in its own plane, are considered.

Finally, it is also evident that the critical load multipliers κ_{cr} will be influenced by the stretched cable and, once its paths is fixed, it will be a function of τ^*_0. More precisely, it will be $\kappa_{cr}(\tau^*_0) \geqslant \kappa_{cr}(0)$ for stabilising paths and $\kappa_{cr}(\tau^*_0) \leqslant \kappa_{cr}(0)$ for the opposite case.

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APPENDIX

Components of the stiffness matrix of eqn (44)

$$\mathbf{K}_{\Gamma\Gamma} = \begin{bmatrix} GA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EA \end{bmatrix} \quad \mathbf{K}_{\Gamma\varphi'} = \mathbf{K}^T_{\varphi'\Gamma} = G\begin{bmatrix} 0 & 0 & Ax_{s2} \\ 0 & 0 & -Ax_{s1} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}_{\Gamma\chi} = \mathbf{K}^T_{\chi\Gamma} = G\begin{bmatrix} C_{01} & C_{11} & C_{21} & C_{31} \\ C_{02} & C_{12} & C_{22} & C_{32} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K}_{\varphi'\varphi'} = \begin{bmatrix} EI_1 & 0 & 0 \\ 0 & EI_2 & 0 \\ 0 & 0 & G(J-D_{s0}) \end{bmatrix} \quad \mathbf{K}_{\varphi'\chi} = \mathbf{K}^T_{\chi\varphi'} = G\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{s0} & C_{s1} & C_{s2} & C_{s3} \end{bmatrix}$$

$$\mathbf{K}_{\chi'\chi'} = G\begin{bmatrix} C_{\psi_0} & C_{\psi_0\psi_1} & C_{\psi_0\psi_2} & C_{\psi_0\psi_3} \\ C_{\psi_1\psi_0} & C_{\psi_1} & C_{\psi_1\psi_2} & C_{\psi_1\psi_2} \\ C_{\psi_2\psi_0} & C_{\psi_2\psi_1} & C_{\psi_2} & C_{\psi_2\psi_3} \\ C_{\psi_3\psi_0} & C_{\psi_3\psi_1} & C_{\psi_3\psi_2} & C_{\psi_3} \end{bmatrix} \quad \mathbf{K}_{\chi'\chi'} = E\begin{bmatrix} I_{\psi_0} & I_{\psi_0\psi_1} & I_{\psi_0\psi_2} & I_{\psi_0\psi_3} \\ I_{\psi_1\psi_0} & I_{\psi_1} & I_{\psi_1\psi_2} & I_{\psi_1\psi_2} \\ I_{\psi_2\psi_0} & I_{\psi_2\psi_1} & I_{\psi_2} & I_{\psi_2\psi_3} \\ I_{\psi_3\psi_0} & I_{\psi_3\psi_1} & I_{\psi_3\psi_2} & I_{\psi_3} \end{bmatrix}$$